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INVARIANT OPERATORS AND UNIVALENT FUNCTIONS[1]

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ABSTRACT. Necessary and sufficient conditions for univalence of meromorphic functions in certain domains in the complex plane are established in terms of some differential operators of degrees $\geqslant 3$, possessing the same invariance property as the Schwarzian derivative. Those operators include the derivatives of the Schwarzian derivative and Aharonov's invariants. Conditions for the existence of quasiconformal extensions are also achieved.

I. Introduction. Let $\mathcal{A}(D)$ and $\mathfrak{M}(D)$ denote the linear spaces of all the analytic and meromorphic functions, respectively, at a domain D in the complete complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, and let $\mathfrak{M}_0(D)$ be the set of all univalent elements in $\mathfrak{M}(D)$. For every positive integer n let

$$(1.1^*) \quad B_n^*(D) = \left\{\phi \in \mathcal{A}(D) \colon \|\phi\|_{n,D}^* = \|\phi\|_n^* = \sup_{z \in D} d(z, \partial D)^n |\phi(z)| < \infty\right\}$$

where $d(z, \partial D)$ is the euclidean distance of z to the boundary ∂D of D. If D is a hyperbolic domain, with a hyperbolic density $\rho_D(z) = \rho(z)$, let

$$(1.1) \quad B_n(D) = \left\{\phi \in \mathcal{A}(D) \colon \|\phi\|_{n,D} = \|\phi\|_n = \sup_{z \in D} \rho(z)^{-n} |\phi(z)| < \infty\right\}.$$

(Here we consider the hyperbolic density $\rho_D(z)$ with constant curvature -4, so that $\rho_U(z) = (1 - |z|^2)^{-1}$ for the unit disc U.)

The Schwarzian differential operator

$$\mathfrak{M}(D) \ni f \mapsto \{f, z\} = \left[\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right](z) \in \mathfrak{M}(D)$$

is a typical invariant operator in the sense that

$$(1.2) \qquad \{g \circ f, z\} = \{f, z\} \quad \text{for every Möbius transformation } g,$$

and it has the following two remarkable properties:

(i) If $f \in \mathfrak{M}_0(D)$ and $\phi(z) = \{f, z\}$, then $\phi \in B_2^*(D)$ and

$$(1.3) \qquad \|\phi\|_{2,D}^* \leqslant 6$$

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(Gehring [7]), and if f has a μ-quasiconformal extension into $\hat{\mathbf{C}} \setminus D$, then

$$(1.3') \qquad \|\phi\|^*_{2,D} \leq 6\|\mu\|_\infty .$$

(ii) Conversely, if ∂D is a finite disjoint union of points and quasicircles, and if $\phi \in B_2(D)$ with

$$(1.4) \qquad \|\phi\|_{2,D} < \varepsilon$$

for some $\varepsilon = \varepsilon(\partial D) > 0$, then $\phi(z) = \{f, z\}$ in D, for some quasiconformal automorphism f of $\hat{\mathbf{C}}$ which is conformal in D (Osgood [14]). In particular if D is the unit disc U, then $\varepsilon(\partial U) = 2$ (Nehari [13], Ahlfors-Weill [4]).

Here we show that there is a large class of differential operators τ: $\mathfrak{M}(D) \to \mathfrak{M}(D)$ which are invariant in the sense that

$$(1.2') \qquad \tau(g \circ f) = \tau(f) \quad \text{for every Möbius transformation } g$$

and which possess properties similar to (i) and (ii), i.e., in terms of which we can formulate criteria for univalence and for quasiconformal extendability of meromorphic functions.

II. Homogeneous invariant operators—definitions. A differential operator τ in $\mathfrak{M}(D)$ is called *invariant* if it satisfies condition (1.2′). Let ϕ_n, $n = 2, 3, 4, \ldots$, denote the differential operators

$$(2.1) \qquad \phi_n(f, z) = \frac{1}{(n+1)!}\{f, z\}^{(n-2)}, \qquad f \in \mathfrak{M}(D), z \in D.$$

Evidently each ϕ_n is invariant (see Lavie [10]) and

$$(2.1') \qquad \begin{aligned} \phi_2(f, z) &= \frac{1}{6}\{f, z\}, \\ \phi_n(f, z) &= \frac{1}{n+1}\phi'_{n-1}(f, z) = \frac{3!}{(n+1)!}\phi_2^{(n-2)}(f, z), \qquad n \geq 3. \end{aligned}$$

Another example of a sequence of invariant operators are the so-called Aharonov operators ψ_n, $n = 2, 3, \ldots$, which are defined in [1] by means of the generating function

$$(2.2) \qquad F(f; z, \zeta) = \frac{f'(z)}{f(z) - f(\zeta)} - \frac{1}{z - \zeta} = \sum_{n=1}^{\infty} \psi_n(f, z)(\zeta - z)^{n-1}, \qquad z \in D, |\zeta - z| < d(z, \partial D).$$

From this definition Aharonov derived the recursion formula:

$$(2.2') \qquad \begin{aligned} \psi_2(f, z) &= \tfrac{1}{6}\{f, z\} = \phi_2(f, z), \\ \psi_n(f, z) &= \frac{1}{n+1}\left[\psi'_{n-1}(f, z) + \sum_{j=2}^{n-2} \psi_j(f, z)\psi_{n-j}(f, z)\right], \qquad n \geq 3. \end{aligned}$$

An operator τ_n in $\mathfrak{M}(D)$ is called *homogeneous* (of degree n) if it has the form

$$(2.3) \qquad \tau_n(f, z) = \sum_{m=1}^{[n/2]} \sum_{k_j \geq 2, \sum_{j=1}^m k_j = n} a_{k_1, k_2, \ldots, k_m} \prod_{j=1}^{m} \phi_{k_j}(f, z), \qquad n \geq 2,$$

for some constants $a_{k_1,k_2,\ldots,k_m}$, $1 \leqslant m \leqslant [\frac{n}{2}]$. A homogeneous operator τ_n is called *normal* if $a_n = 1$. Note that every normal homogeneous operator τ_n of degree n is an invariant differential operator of order $(n+1)$. The operators ϕ_n, $n = 2, 3, 4, \ldots$, are the simplest example for normal homogeneous operators. On the other hand formula (2.2′) implies, by induction, that each ψ_n is also normal homogeneous.

III. Necessary conditions for univalence. In this section we prove a necessary condition for univalence and for quasiconformal extendability of $f \in \mathfrak{M}_0(D)$ in terms of every homogeneous operator τ_n: $\mathfrak{M}(D) \to \mathfrak{M}(D)$. The proof is based on the following three lemmas:

LEMMA 1. *If $f \in \mathfrak{M}_0(U)$, where U is the unit disc, then*

$$\sum_{n=2}^{\infty} (n-1)\,|\psi_n(f,0)|^2 \leqslant 1 \tag{3.1}$$

and if f has a μ-quasiconformal extension into $\hat{\mathbf{C}} \setminus U$, then

$$\sum_{n=2}^{\infty} (n-1)\,|\psi_n(f,0)|^2 \leqslant \|\mu\|_\infty^2. \tag{3.1′}$$

PROOF. Aharonov proved (3.1) by showing that it is a new formulation of the classical area theorem. Similarly (3.1′) is just another form of Lehto's area theorem for univalent functions with quasiconformal extension [**11**].

LEMMA 2. *If $f \in \mathfrak{M}_0(D)$ for any domain D, then*

$$\sum_{n=2}^{\infty} (n-1)d(z,\partial D)^{2n}\,|\psi_n(f,z)|^2 \leqslant 1, \qquad z \in D, \tag{3.2}$$

and if f has a μ-quasiconformal extension into $\hat{\mathbf{C}} \setminus D$, then

$$\sum_{n=2}^{\infty} (n-1)d(z,\partial D)^{2n}\,|\psi_n(f,z)|^2 \leqslant \|\mu\|_\infty^2. \tag{3.2′}$$

PROOF. Fix any point $z \in D$ and set $h(\zeta) = f(r\zeta + z)$ where $r = d(z, \partial D)$. From (2.2) we conclude

$$\psi_n(h,\zeta) = r^n\psi_n(f, r\zeta + z), \qquad |\zeta| < 1. \tag{3.3}$$

Hence

$$\psi_n(h,0) = r^n\psi_n(f,z). \tag{3.3′}$$

Now, if $f \in \mathfrak{M}_0(D)$ then $h \in \mathfrak{M}_0(U)$, and if f has a μ-quasiconformal extension into $\hat{\mathbf{C}} \setminus D$, h has a $\tilde{\mu}$-quasiconformal extension into $\hat{\mathbf{C}} \setminus U$, with $\|\tilde{\mu}\|_\infty = \|\mu\|_\infty$. The lemma follows by setting h instead of f in Lemma 1 and applying (3.3′).

LEMMA 3. *For every domain D, the differentiation operators d^l: $B_n^*(D) \ni \phi \mapsto \phi^{(l)} \in B_{n+l}^*(D)$, $l = 0, 1, 2, \ldots$, $n \geqslant 1$, are bounded by constants $K(l, n)$ depending only on l and n.*

PROOF. Let $\phi \in \mathcal{Q}(D)$ with

$$|\phi(z)| \leqslant Cd(z,\partial D)^{-n}, \qquad C = \|\phi\|_n^*,\ z \in D,\ n \geqslant 1. \tag{3.4}$$

Fix a point $z \in D$ and let $R = d(z, \partial D)$, $\Gamma_r = \{\zeta: |\zeta - z| = r\}$ for $0 < r < R$. Cauchy's inequalities and (3.4) imply:

$$|\phi^{(l)}(z)| \leq \frac{l!}{r^l} \max_{\zeta \in \Gamma_r} |\phi(\zeta)| \leq Cl! r^{-l} \max_{\zeta \in \Gamma_r} d(\zeta, \partial D)^{-n} = Cl! r^{-l}(R - r)^{-n}.$$

Hence

$$|\phi^{(l)}(z)| \leq Cl! \min_{0<r<R} r^{-l}(R - r)^{-n} = Cl! \left(\frac{n+l}{n}\right)^n \left(\frac{n+l}{l}\right)^l R^{-n-l}. \quad \text{Q.E.D.}$$

THEOREM 1. *Every homogeneous operator τ_n has a positive constant $K^*(\tau_n) = K_n^*$, which depends only on τ_n, such that*

(i) *if $f \in \mathfrak{M}_0(D)$, for any domain D, and $\phi = \tau_n(f, \cdot)$, then*

$$\|\phi\|_{n,D}^* \leq K_n^*; \tag{3.5*}$$

(ii) *if f has a μ-quasiconformal extension into $\hat{\mathbf{C}} \setminus D$, then*

$$\|\phi\|_{n,D}^* \leq K_n^* \|\mu\|_\infty, \tag{3.5**}$$

(iii)

$$K^*(\psi_n) \leq 1/\sqrt{n-1}, \qquad n \geq 2.$$

PROOF. For $\tau_n = \psi_n$ the theorem follows from Lemma 2. In particular $K^*(\psi_2) = 1$ (cf. Gehring [7, Corollary 1, p. 563]). Next, Lemma 3 implies

$$\|\phi_n(f)\|_n^* = \frac{3!}{(n+1)!} \|\phi_2^{(n-2)}(f)\|_n^* \leq \frac{3!}{(n+1)!} K(n-2, 2) \|\phi_2(f)\|_2^*.$$

Thus, (3.5*) and (3.5**) for ϕ_n follow from the corresponding inequalities for $\phi_2 = \psi_2$, i.e.

$$K^*(\phi_n) \leq \frac{3!}{(n+1)!} K(n-2, 2) K^*(\phi_2) = \frac{3!}{(n+1)!} K(n-2, 2).$$

Finally, for the general homogeneous invariant operator τ_n (3.5*) and (3.5**) follow from (2.3) and the obvious inequality

$$\left\| \prod_{j=1}^m \phi_{k_j}(f) \right\|_n^* \leq \prod_{j=1}^m \|\phi_{k_j}(f)\|_{k_j}^*, \qquad \sum_{j=1}^m k_j = n. \quad \text{Q.E.D.}$$

REMARK. Schwarz' lemma implies

$$\rho_D(z) d(z, \partial D) \leq 1, \qquad z \in D, \tag{3.6}$$

for every hyperbolic domain D. Therefore it follows from (1.1*) and (1.1) that $B_n(D) \subseteq B_n^*(D)$. Conversely if D is also simply-connected and $\infty \notin D$, then Koebe's $\frac{1}{4}$-theorem yields

$$\rho_D(z) d(z, \partial D) \geq 1/4, \qquad z \in D. \tag{3.7}$$

Therefore for such domains $B_n(D) = B_n^*(D)$ with equivalent norms. Hence, for such domains one can omit the asterisks in Theorem 1 and obtain

COROLLARY 1. *If D is a hyperbolic simply-connected domain in $\mathbf{C}$, then for every homogeneous operator τ_n and every $f \in \mathfrak{M}_0(D)$ the function $\phi(z) = \tau_n(f, z)$ satisfies*

$$\|\phi\|_{n,D} \leq K(\tau_n) \tag{3.5'}$$

where the constant $K(\tau_n)$ depends only on τ_n. Furthermore, if f has a μ-quasiconformal extension into $\hat{\mathbf{C}} \setminus D$, then

$$\|\phi\|_{n,D} \leq K(\tau_n)\|\mu\|_\infty. \tag{3.5''}$$

IV. Sufficient conditions. We now prove the following generalization of property (ii) (see Introduction) of the Schwarzian derivative for every *normal* homogeneous operator.

THEOREM 2. *Let C be either a circle in $\mathbf{C}$ or a quasicircle passing through ∞, let D_1 and D_2 be the components of $\hat{\mathbf{C}} \setminus C$ such that $\infty \notin D_2$ if C is a circle. Then every normal homogeneous operator τ_n in $\mathfrak{M}(D_2)$ has a positive constant $\varepsilon_n = \varepsilon(C, \tau_n)$, depending only on C and τ_n, such that for any $\phi \in B_n(D_2)$ with norm $\|\phi\|_n < \varepsilon_n$ there is a quasiconformal automorphism f of $\hat{\mathbf{C}}$ which is conformal in D_2 and $\tau_n(f, z) = \phi(z)$ in D_2.*

SKETCH OF PROOF. In order to prove this theorem one should find, for every small enough $\phi \in B_n(D_2)$, a univalent meromorphic solution $f(z)$ in D_2 for the differential equation $\tau_n(f, z) = \phi(z)$ of order $(n + 1)$, and a quasiconformal extension of $f(z)$ into $D_1 = \hat{\mathbf{C}} \setminus \overline{D}_2$. In other words one would like to have, near the origin in $B_n(D_2)$, a local cross-section for the following nonlinear mapping:

$$\mathfrak{T}_c^{(n)}: M(D_1) \ni \mu \to \tau_n^\mu = \tau_n(f_\mu) \in B_n(D_2) \tag{4.1}$$

where $M(D_1)$ is the set of all Beltrami coefficients μ supported in D_1 (i.e., $M(D_1)$ is the open unit ball in the complex Banach space $L^\infty(D_1)$ of all the bounded-measurable complex-valued functions supported in D_1) and f_μ is the unique μ-quasiconformal automorphism of $\hat{\mathbf{C}}$ which fixes the points $\{0, 1, \infty\}$ (this normalization causes no loss of generality because of the invariance property of τ_n and since every μ-quasiconformal automorphism of $\hat{\mathbf{C}}$ differs from f_μ by a left composition with a Möbius transformation). Ahlfors-Weill [4] gave such a local cross-section for $\tau_2 = \phi_2$ and D_2 is the unit disc U. In the first part of our proof we use their result to give a cross-section for $\tau_n = \phi_n$ and $D_2 = U$ (Lemma 4). In the second part of the proof we show that the mapping $\mathfrak{T}_c^{(n)}$ of (4.1) is holomorphic, and we compute its derivative $D\mathfrak{T}_c^{(n)}(0)$ at the origin (Lemma 5). Next we find a continuous global cross-section for the continuous linear mapping $D\mathfrak{T}_c^{(n)}(0)$: $L^\infty(D_1) \to B_n(D_2)$ whenever C is either a quasicircle passing through ∞ (Lemma 6) or a circle (Lemma 6′). Thus the theorem is finally established by applying the inverse mapping theorem.

LEMMA 4. *Every $\phi \in B_{n+1}(U)$ has a $\psi \in B_n(U)$ such that $\psi' = \phi$ and $\|\psi\|_n \leq (2^n/n)\|\phi\|_{n+1}$.*

PROOF. For every $\phi \in B_{n+1}(U)$ we simply define

$$\psi(z) = \int_0^z \phi(\zeta)\, d\zeta = \int_0^1 \phi(tz) z\, dt, \qquad |z| < 1.$$

Obviously $\psi'(z) = \phi(z)$ in U and

$$\begin{aligned}\left(1 - |z|^2\right)^n |\psi(z)| &\leq \int_0^1 \frac{\|\phi\|_{n+1}\left(1 - |z|^2\right)^n}{\left(1 - t|z|\right)^{n+1}} |z|\, dt \\ &= \frac{\left(1 + |z|\right)^n}{n}\left[1 - \left(1 - |z|\right)^n\right] \|\phi\|_{n+1} \leq \frac{2^n}{n} \|\phi\|_{n+1}, \qquad |z| < 1,\end{aligned}$$

which proves the lemma.

Thus, if $\phi \in B_n(U)$ and

$$\|\phi\|_n \leq \frac{1}{n(n+1)} 2^{1-(n+1)(n-2)/2} \tag{4.2}$$

then by successive use of Lemma 4 one can obtain a function $\psi \in B_2(U)$ such that $\phi(z) = \psi^{(n-2)}(z)/(n+1)!$, $|z| < 1$, and

$$\|\psi\|_2 \leq \frac{2^{2+3+\cdots+(n-1)}}{(n-1)!} \|\psi^{(n-2)}\|_n < 2.$$

Hence, by Ahlfors-Weill [**4**, Theorem A], $\psi(z) = \{f, z\}$, $z \in U$, where f is a quasiconformal automorphism of $\hat{\mathbf{C}}$ which is conformal in U, and $\phi(z) = \{f, z\}^{(n-2)}/(n+1)! = \phi_n(f, z)$. This completes the first part of the proof of Theorem 2.

In the next lemma we show that infinitesimally all the mappings $\mathfrak{T}_c^{(n)}$, defined in (4.1), behave the same as the mapping

$$\Phi_c^{(n)}: M(D_1) \ni \mu \mapsto \phi_n^\mu = \phi_n(f_\mu, \cdot) \in B_n(D_2) \tag{4.1'}$$

near the center of $M(D_1)$.

LEMMA 5. *The mapping $\mathfrak{T}_c^{(n)}$, defined in* (4.1), *is a holomorphic function from $M(D_1)$ into $B_n(D_2)$, and its derivative at the origin is the continuous linear mapping*

$$D\mathfrak{T}_c^{(n)}(0) = \mathfrak{M}_c^{(n)}: L^\infty(D_1) \ni \mu \mapsto -\frac{1}{\pi} \iint_{D_1} \frac{\mu(\zeta)}{(\zeta - z)^{n+2}} d\xi\, d\eta, \qquad z \in D_2,\ \zeta = \xi + i\eta. \tag{4.3}$$

PROOF. Ahlfors and Bers [**3**] proved that $f_{t\mu}(z)$ is an analytic function of t for every z and $\mu \in M(D_1)$ when t is small enough, and they showed that

$$\frac{\partial f_{t\mu}(z)}{\partial t}\Big|_{t=0} = F_\mu(z) = -\frac{1}{\pi} \iint_{D_1} \left(\frac{1}{\zeta - z} - \frac{z}{\zeta - 1} + \frac{z-1}{\zeta}\right) \mu(\zeta)\, d\xi\, d\eta, \qquad z \in \mathbf{C}. \tag{4.4}$$

Later Bers proved in [6] that the mapping $\Phi_c^{(2)}\colon M(D_1) \to B_2(D_2)$ is holomorphic with the derivative

$$(4.5)\qquad \begin{aligned} D\Phi_c^{(2)}(0)\colon \mu \mapsto \frac{\partial \phi_2^{t\mu}(z)}{\partial t}\Big|_{t=0} &= \left(\mathfrak{M}_c^{(2)}\mu\right)(z) = \frac{1}{3!}F_\mu^{(3)}(z) \\ &= -\frac{1}{\pi}\int\!\!\int_{D_1} \frac{\mu(\zeta)}{(\zeta - z)^4}\, d\xi\, d\eta, \qquad z \in D_2. \end{aligned}$$

But by (2.1′) we have

$$(4.6)\qquad \Phi_c^{(n)} = \frac{3!}{(n+1)!} d^{(n-2)} \circ \Phi_c^{(2)}.$$

Hence, by the chain rule, $\Phi_c^{(n)}$ is holomorphic and

$$(4.7)\qquad D\Phi_c^{(n)}(0) = \frac{3!}{(n+1)!} d^{(n-2)} \circ D\Phi_c^{(2)}(0)$$

and therefore (4.5) implies

$$(4.8)\qquad \begin{aligned} D\Phi_c^{(n)}(0) &= \frac{3!}{(n+1)!} d^{n-2} \circ \mathfrak{M}_c^{(2)}\colon \mu \mapsto \left(\mathfrak{M}_c^{(n)}\mu\right)(z) \\ &= \frac{1}{(n+1)!} F_\mu^{(n+1)}(z) = -\frac{1}{\pi}\int\!\!\int_{D_1} \frac{\mu(\zeta)}{(\zeta - z)^{n+2}}\, d\xi\, d\eta. \end{aligned}$$

Now, notice that the following product of holomorphic mappings

$$\prod_{j=1}^{m} \Phi_{k_j}\colon M(D_1) \to B_n(D_2), \qquad \sum_{j=1}^{m} k_j = n$$

is holomorphic, and therefore $\mathfrak{T}_c^{(n)}$ is holomorphic. Hence

$$D\mathfrak{T}_c^{(n)}(0)[\mu] = \lim_{t\to 0} \frac{1}{t}\left(\mathfrak{T}_c^{(n)}(t\mu) - \mathfrak{T}_c^{(n)}(0)\right) = \lim_{t \to 0} \frac{\tau_n^{t\mu} - \tau_n^0}{t} = \frac{\partial \tau_n^{t\mu}}{\partial t}\Big|_{t=0}$$

and since $\phi_{k_j}^0(z) = \phi_{k_j}(\mathrm{id}, z) \equiv 0$, Leibnitz rule and (2.3) yield

$$D\mathfrak{T}_c^{(n)}(0)[\mu] = \frac{\partial \tau_n^{t\mu}}{\partial t}\Big|_{t=0} = \frac{\partial \phi_n^{t\mu}}{\partial t}\Big|_{t=0} = D\Phi_c^{(n)}(0)[\mu] = \mathfrak{M}_c^{(n)}\mu.$$

This implies (4.3). Finally, for every $z \in D_2$ we get

$$\begin{aligned} \left|\left(\mathfrak{M}_c^{(n)}\mu\right)(z)\right| &\leqslant \frac{\|\mu\|_\infty}{\pi}\int\!\!\int_{D_1} \frac{d\xi\, d\eta}{|\zeta - z|^{n+2}} \\ &\leqslant \frac{\|\mu\|_\infty}{\pi}\int\!\!\int_{|\zeta - z| > d(z,C)} |\zeta - z|^{-n-2}\, d\xi\, d\eta \\ &= \frac{2}{n} d(z, \partial D_1)^{-n} \|\mu\|_\infty, \qquad \mu \in L^\infty(D_1). \end{aligned}$$

Hence, by (3.7) we conclude that $\mathfrak{M}_c^{(n)}\colon L^\infty(D_1) \to B_n(D_2)$ is bounded, and this completes the proof of Lemma 5.

Ahlfors proved in [2, Lemma 2] that every quasicircle C passing through ∞ admits a quasiconformal reflection h (i.e. an involution $h\colon \mathbf{C} \to \mathbf{C}$ such that $h|_c = \mathrm{id}$ and $z \to \overline{h(z)}$ is quasiconformal) which is uniformly bilipschizian in $\mathbf{C}$, i.e.:

$$(4.9)\qquad K^{-1}|z_1 - z_2| \leqslant |h(z_1) - h(z_2)| \leqslant K|z_1 - z_2|$$

for some constant $K = K(C) > 1$ and all $z_1, z_2 \in \mathbf{C}$. For $z_2 \in C$ we have in particular

$$(4.9')\qquad |z_1 - z_2| \leqslant K|h(z_1) - z_2|.$$

LEMMA 6. *Let C be a quasicircle passing through ∞, and h a bilipschizian quasiconformal reflection at C. Then the linear mapping*

$$(4.10)\qquad \Lambda_c^{(n)}\colon B_n(D_2) \ni \psi \mapsto (n+1)(\zeta - h(\zeta))^n \frac{\partial h(\zeta)}{\partial \bar{\zeta}} \psi(h(\zeta)) \in L^\infty(D_1), \qquad \zeta \in D_1,$$

is a continuous right-inverse of $\mathfrak{M}_c^{(n)}\colon L^\infty(D_1) \to B_n(D_2)$.

PROOF. We will show that the mapping $\Lambda_c^{(n)}$ is indeed from $B_n(D_2)$ into $L^\infty(D_1)$, is bounded and satisfies the reproduction formula

$$(4.11)\qquad \psi(z) = \left(\mathfrak{M}_c^{(n)} \circ \Lambda_c^{(n)}\psi\right)(z) = -\frac{1}{\pi}\iint_{D_1} \frac{(\Lambda_c^{(n)}\psi)(\zeta)}{(\zeta - z)^{n+2}}\, d\xi\, d\eta, \qquad z \in D_2, \psi \in B_n(D_2).$$

Observe, first, that according to formulas (4.8) and (4.4)

$$(4.12)\qquad \begin{aligned} \left(\mathfrak{M}_c^{(n)}\mu\right)(z) &= \frac{F_\mu^{(n+1)}(z)}{(n+1)!} \\ &= \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left\{-\frac{z(z-1)}{\pi}\iint_{D_1} \frac{\mu(\zeta)\, d\xi\, d\eta}{(\zeta - z)\zeta(\zeta - 1)}\right\}, \qquad z \in D_2, \end{aligned}$$

where $F_\mu(z)$ is known [8, pp. 129, 136] as the unique continuous solution in C of the nonhomogeneous Cauchy-Riemann equation

$$(4.13)\qquad \partial F(z)/\partial \bar{z} = \mu(z)$$

which satisfies $F_\mu(0) = F_\mu(1) = 0$ and

$$(4.14)\qquad F_\mu(z) = O(|z|\log|z|) \quad \text{as } z \to \infty.$$

But also if $\tilde{F}_\mu(z)$ is any other continuous solution of (4.13) which satisfies only

$$(4.14')\qquad \tilde{F}_\mu(z) = O\left(|z|^2\right) \quad \text{as } z \to \infty$$

then by Weyl's lemma and Liouville's theorem $\tilde{F}_\mu - F_\mu$ is a polynomial of degree $\leqslant 2$, and hence

$$(4.12')\qquad \left(\mathfrak{M}_c^{(n)}\mu\right)(z) = \frac{\tilde{F}_\mu^{(n+1)}(z)}{(n+1)!}, \qquad z \in D_2, n \geqslant 2.$$

Thus, in order to reproduce a Beltrami differential $\mu = \Lambda_c^{(n)}\psi \in L^\infty(D_1)$ with $\mathfrak{M}_c^{(n)}\mu = \psi$ for every given $\psi \in B_n(D_2)$, one should, first, choose an analytic solution $F(z)$ in D_2 of the equation

$$F^{(n+1)}(z) = (n+1)!\psi(z), \qquad z \in D_2. \tag{4.15}$$

Next one has to find an extension of this $F(z)$ into $D_1 = \hat{\mathbf{C}} \setminus \overline{D_2}$ which would have a generalized derivative $\mu = \partial F/\partial \bar{z} \in L^\infty(D_1)$ with a L^∞-norm comparable to $\|\psi\|_{n,D_2}$.

We assume, first, that ψ belongs to the following subspace of $B_n(D_2)$:

$$\tilde{B}_n(D_2) = \{\psi \in B_n(D_2) : \psi = \hat{\psi}|_{D_2} \text{ where } \hat{\psi} \in B_n(D_0) \text{ for some } D_0 = D_0(\psi) \supset \overline{D_2}\}.$$

Then the solution $F(z)$ of (4.15), for $\psi \in \tilde{B}_n(D_2)$, has a continuous extension to $\overline{D_2} = D_2 \cup C$. Moreover, we have

$$|\psi(z)| = |\hat{\psi}(z)| \leq \|\hat{\psi}\|_{n,D_0} \cdot \rho_{D_0}(z)^n = O(|z|^{-2n}) \quad \text{as } z \to \infty \text{ from } D_2,$$

since $\infty \in C \subset D_0$ and $\rho_{D_0}(z) = O(|z|^{-2})$ as $z \to \infty$. Hence we can choose a solution $F(z)$ of (4.15) such that

$$F(z) = O(|z|^{-2n+n+1}) = O(|z|^2) \quad \text{as } z \to \infty. \tag{4.14*}$$

Now, following Bers' method in [6] we extend $F(z)$ as follows:

$$\hat{F}(z) = \begin{cases} F(z), & z \in D_2 \cup C, \\ \displaystyle\sum_{k=0}^{n} \frac{(z - h(z))^k}{k!} F^{(k)}(h(z)), & z \in D_1. \end{cases}$$

Obviously $\hat{F}(z)$ is well defined and continuous in $\mathbf{C}$. Moreover, by the chain rule, (4.9) and (4.15), the generalized derivative $\partial \hat{F}/\partial \bar{z}$ exists almost everywhere and we have

$$\frac{\partial \hat{F}}{\partial \bar{z}} = \frac{(z - h(z))^n}{n!} \frac{\partial h(z)}{\partial \bar{z}} F^{(n+1)}(h(z)) = (\Lambda_c^{(n)}\psi)(z), \qquad z \in D_1.$$

Now we show that $\mu = \Lambda_c^{(n)}\psi \in L^\infty(D_1)$ for every $\psi \in B_n(D_2)$. Indeed, by (4.9′) we have for every $z \in D_1$ and $z_0 \in C$:

$$|z - h(z)| \leq |z - z_0| + |h(z) - z_0| \leq (K+1)|h(z) - z_0|.$$

Hence, by (3.6) $|z - h(z)| \leq (K+1)d(h(z), C) \leq (K+1)\rho_{D_2}(h(z))^{-1}$. Observe also that (4.9) implies that the generalized derivative $\partial h(z)/\partial \bar{z}$ is essentially bounded by K, and therefore

$$\begin{aligned} |(\Lambda_c^{(n)}\psi)(z)| &\leq (n+1)(K+1)^n K \rho_{D_2}(h(z))^{-n} |\psi(h(z))| \\ &\leq (n+1)(K+1)^n K \|\psi\|_{n,D_2}, \qquad z \in D_1, \end{aligned}$$

which proves also the continuity of the linear mapping $\Lambda_c^{(n)}: B_n(D_2) \to L^\infty(D_1)$. Finally, the above construction yields

$$(\mathfrak{M}_c^{(n)} \circ \Lambda_c^{(n)}\psi)(z) = (\mathfrak{M}_c^{(n)}\mu)(z) = \frac{\hat{F}^{(n+1)}(z)}{(n+1)!} = \frac{F^{(n+1)}(z)}{(n+1)!} = \psi(z),$$

$$z \in D_2, \psi \in \tilde{B}_n(D_2),$$

i.e., every $\psi \in \tilde{B}_n(D_2)$ satisfies the required reproduction formula

$$(4.11') \quad \psi(z) = -\frac{n+1}{\pi}\iint_{D_1} \frac{(\zeta - h(\zeta))^n}{(\zeta - z)^{n+2}} \frac{\partial h(\zeta)}{\partial \bar{\zeta}} \psi(h(\zeta))\, d\xi\, d\eta, \qquad z \in D_2.$$

Suppose now that $\psi \in B_n(D_2)$. Then by [**6**, Lemma 1] there is a sequence $\{\psi_j\}$ in $\tilde{B}_n(D_2)$ which pointwise converges to ψ in D_2, and such that $\|\psi_j\|_n \leqslant \|\psi\|_n$, $j = 1, 2, \ldots$. Then $\{\Lambda_c^{(n)}\psi_j\}_{j=1}^{\infty}$ converges to $\Lambda_c^{(n)}\psi$ pointwise in D_1, and is bounded in $L^\infty(D_1)$, since $\Lambda_c^{(n)}$: $B_n(D_2) \to L^\infty(D_1)$ is bounded. Hence, Lebesgue's bounded convergence theorem yields

$$\begin{array}{ccc} \psi_j(z) & = & -\dfrac{1}{\pi}\displaystyle\iint_{D_1} \frac{(\Lambda_c^{(n)}\psi_j)(\zeta)}{(\zeta - z)^{n+2}}\, d\xi\, d\eta \\ \downarrow & & \downarrow \qquad\qquad (z \in D_2) \\ \psi(z) & = & -\dfrac{1}{\pi}\displaystyle\iint_{D_1} \frac{(\Lambda_c^{(n)}\psi)(\zeta)}{(\zeta - z)^{n+2}}\, d\xi\, d\eta \end{array}$$

which completes the proof of Lemma 6.

Lemma 6 implies the following analogue of Lemma 4:

LEMMA 4′. *Let C be a quasicircle passing through ∞, and $l \geqslant n \geqslant 2$. Then there is a constant $K = K(C, l, n)$, depending only on C, l and n, such that for every $\psi \in B_l(D_2)$ there is a function $\phi \in B_n(D_2)$ with $\phi^{(l-n)} = \psi$ and $\|\phi\|_n \leqslant K\|\psi\|_l$.*

PROOF. Formula (4.8) implies

$$\mathfrak{M}_c^{(l)} = \frac{(n+1)!}{(l+1)!} d^{l-n} \circ \mathfrak{M}_c^{(n)}, \qquad l \geqslant n \geqslant 2,$$

and since $\mathfrak{M}_c^{(l)} \circ \Lambda_c^{(l)} = \mathrm{id}_{B_l(D_2)}$, it follows that the continuous linear mapping $((n+1)!/(l+1)!)\mathfrak{M}_c^{(n)} \circ \Lambda_c^{(l)}$: $B_l(D_2) \to B_n(D_2)$ is a right inverse of the differentiation mapping d^{l-n}: $B_n(D_2) \to B_l(D_2)$, which proves the lemma.

LEMMA 6′. *Let D_2 be a disc in $\mathbf{C}$. For every integer $n \geqslant 2$ the mapping $\mathfrak{M}_c^{(n)}$: $L^\infty(D_1) \to B_n(D_2)$ has a continuous cross-section. In particular, for $D_2 = U$ and $n = 2$, the linear mapping*

$$(4.10') \quad B_2(U) \ni \psi \mapsto (\Lambda_c^{(2)}\psi)(\zeta) = -3\frac{(|\zeta|^2 - 1)^2}{\bar{\zeta}^4}\psi\left(\frac{1}{\bar{\zeta}}\right) \in L^\infty(\hat{\mathbf{C}} \setminus U), \qquad |\zeta| > 1,$$

is a continuous right-inverse of the mapping $\mathfrak{M}_c^{(2)}$: $L^\infty(\hat{\mathbf{C}} \setminus U) \to B_2(U)$.

PROOF. It is enough to prove the lemma for the unit disc U. Ahlfors-Weill proved in [**4**] that the bounded linear mapping

$$\{\phi \in B_2(U): \|\phi\|_2 < 2\} \ni \phi \mapsto \frac{1}{6}(\Lambda_c^{(2)}\phi)(\zeta) \in M(\hat{\mathbf{C}} \setminus U), \qquad |\zeta| > 1,$$

is a local cross-section of the holomorphic mapping $M(\hat{\mathbf{C}} \setminus U) \ni \mu \mapsto 6\phi_2^\mu = \{f_\mu, \cdot\} \in B_2(U)$, i.e. $\Phi_c^{(2)} \circ \Lambda_c^{(2)} = \mathrm{id}$ in the ball of radius 2 around the origin in $B_2(U)$.

Hence, by the chain rule and (4.5)

$$D\Phi_c^{(2)}(0) \circ \Lambda_c^{(2)} = \mathfrak{M}_c^{(2)} \circ \Lambda_c^{(2)} = \mathrm{id}_{B_2(U)}.$$

This with Lemma 4 proves the lemma.

We now return to the proof of Theorem 2. By Lemma 5 the mapping $\mathfrak{T}_c^{(n)}$: $M(D_1) \to B_n(D_2)$ is holomorphic with the derivative $\mathfrak{M}_c^{(n)}$: $L^\infty(D_1) \to B_n(D_2)$ at the origin. By Lemmas 6 and 6′ $\mathfrak{M}_c^{(n)}$ has a continuous cross-section $\Lambda_c^{(n)}$: $B_n(D_2) \to L^\infty(D_1)$. Hence, the restriction of $\mathfrak{M}_c^{(n)}$ to the closed subspace Im $\Lambda_c^{(n)}$ of $L^\infty(D_1)$ is a topo-isomorphism of Im $\Lambda_c^{(n)}$ onto $B_n(D_2)$. Therefore, by the inverse mapping theorem there are neighborhoods $\mathfrak{N}_1$ and $\mathfrak{N}_2$ of the origins in $M(D_1) \cap \mathrm{Im}\, \Lambda_c^{(n)}$ and in $B_n(D_2)$, respectively, such that $\mathfrak{T}_c^{(n)}|_{\mathfrak{N}_1}$ is a holomorphic homeomorphism of $\mathfrak{N}_1$ onto $\mathfrak{N}_2$, and in particular $\mathfrak{N}_2 \subseteq \mathrm{Im}\, \mathfrak{T}_c^{(n)}$, i.e. Im $\mathfrak{T}_c^{(n)}$ in $B_n(D_2)$ contains an open ball of a positive radius ε_n around the origin. This completes the proof of the theorem.

REFERENCES

1. D. Aharonov, *A necessary and sufficient condition for univalence of a meromorphic function*, Duke Math. J. **36** (1969), 599–604.
2. L. V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.
3. L. V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 385–404.
4. L. V. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equation*, Proc. Amer. Math. Soc. **13** (1962), 975–978.
5. A. F. Beardon and F. W. Gehring, *Schwarzian derivative, Poincaré metrics and the kernel function*, Comment. Math. Helv. **55** (1980), 50–64.
6. L. Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134.
7. F. W. Gehring, *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv. **52** (1977), 561–572.
8. I. Kra, *Automorphic forms and Kleinian groups*, Benjamin, New York, 1972.
9. W. Kraus, *Über den Zusammengang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreis abbildung*, Mitt. Math. Sem. Giessen **21** (1932), 1–28.
10. M. Lavie, *The Schwarzian derivative and disconjugacy of n-th order linear differential equations*, Canad. J. Math. **21** (1969), 235–249.
11. O. Lehto, *Schlicht functions with a quasiconformal extension*, Ann. Acad. Sci. Fenn. Ser. A I Math. **500** (1971), 1–10.
12. ______, *Conformal mappings and Teichmüller spaces*, Lectures at the Technion, Haifa, Israel, April-May 1973.
13. Z. Nehari, *Schwarzian derivatives and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
14. B. Osgood, *Univalence criteria in multiply connected domains*, Trans. Amer. Math. Soc. **260** (1980), 459–473.

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